https://doi.org/10.46344/JBINO.2025.v14i03.14

COMPARATIVE STUDY OF OPTIMALITY CRITERIA FOR PARTIALLY BALANCED LATTICE DESIGNS

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ABSTRACT

Partially Balanced Lattice Designs (PBLD) is a subclass of Incomplete Block Designs which are similar to the Balanced Lattices only that they allows for more flexible choice of the number of replications, the PBLD requires that the number of treatments must be a perfect square and that Block size k must be equal to the square root of these treatments number. Balanced Lattice Design require the number of replications to be $k + 1$. They exist for certain parameters. They require large number of replications, which consumes logistics, time and effectiveness. The aim of this study is to compare Optimality Criteria for a Partially Balanced Lattice Designs with three associate classes. The design based of A-, D-, and G- optimality criteria were employed. This approach demonstrated in our study involving thirty-two, forty-eight and sixty-four treatments. The results show that D-optimality has the highest values in all the categories followed by G- and A-optimality criteria respectively, it means that D- criteria is more optimal than A- and G- criteria. In the same manner, the efficiencies of this Design were considered by maximizing the information matrix; the results revealed that A- and D- efficiency criteria have the same efficiency and greater than G- criteria in all the categories, D-efficiency Criteria is consider to be the most optimal and most efficient among all the criteria despite having the same efficiencies with A-optimality. Hence, it show that the more, the replication the more the optimal and efficient of the design. It is therefore recommended here that for studies in Partial Lattice Designs, D- Optimality is better. Agricultural researchers, sample surveyors, plant breeders, environmentalists, social scientists as well as medical and health officials should use Partial Lattice Designs to test a large number of entries that are compare directly for selection, it is cost effective in experimental Designs and improve efficiency. It also serves as reference material for researchers who wishes to carryout research on Partial Lattices.

Keywords: Incidence Matrix, Optimality Criteria and Efficiency Criteria.

Introduction

The complete block design types of experiments are inefficient for large number of treatments, because of their failure to adequately minimize the effect of soil heterogeneity (Katsileros. et al., 2015). Generally, the greater the heterogeneity within blocks, the poorer the precision of variety effect estimates. Incomplete block designs are arranged in relatively small blocks that contain fewer varieties than the total number of varieties to be compared.

The designs in which the block phenomenon is followed but the condition of having all the treatments in all blocks is not met are called Incomplete Block designs. In Incomplete Block situations, the use of several small blocks with fewer treatments results in gains in precision but at the expense of a loss of information on comparisons within blocks. Incomplete block designs are now widely used in plant breeding and variety testing around the world. But the analysis of data for incomplete block designs is more complex than complete block design. Thus where computation facilities are limited, incomplete block designs should be considered a last option (Nokoe, 2017).

When the number of treatments is very large and blocking is necessary, Incomplete Block Designs (IBD) is generally used. The origins of Incomplete Block Design dates back to Yates, 1936, who introduced the concept of Balanced Incomplete Block Designs and their analysis utilizing both intra- and inter-block information. He referred to these designs as quasi-factorial or Lattice Designs. In order to eliminate heterogeneity; a concept of Balanced Incomplete Block Designs (BIBD) was introduced (Awad and Banerjee, 2013).

The arrangement of "v" treatments in "b" blocks each of size "k", each treatment appears exactly in "r" blocks and every pair of treatments occurs exactly "λ" times, then the design is said to be Balanced Incomplete Block Design (Shekar and Bhatra, 2016).

Balanced Incomplete Block Designs have several advantages. They are connected designs and the block sizes are equal. A design where all the element contrasts are estimable is a connected design. Otherwise, it is a disconnected design. Another important property of the BIBD is that it is balanced. This means that all the treatments difference is estimated with the same accuracy. A restriction in using the BIBD is that they are not available for all parameter combinations. They exist only for certain parameters. Sometimes, they require large number of replications and this hampers the utility of the BIBD (Salihu*et al.*, 2021).

The PBIB designs belong to the class of incomplete block designs which require lesser experimental material as compared to the complete block designs. The PBIB designs have found their importance in many fields ranging from agriculture experiments, plant breeding, medicine testing, genetic engineering (Sharma and Garg 2022).

Partially Balanced Incomplete Block Designs remain connected like BIBD but are no more balanced, rather they are Partially Balanced in the sense that some pairs of treatments have the same efficiency whereas other pairs have similar efficiency but different from the efficiency of the earlier pairs of treatments. The notion of Partially Balanced Incomplete Block Designs was introduced by Bose and Nair (1939) which encompass some of the Lattice Designs introduced earlier by Yates. Shekharappa*et al.* (2013) noted that PBIBD is an arrangement of v symbols in to b sets (called blocks) of size k, k< v such that

i. Every symbol is contained exactly in r blocks.
ii. Each block contains k distinct symbols.
iii. Any two symbols which are i^{th} associates occur together in λi blocks.

Srisuradetchai. (2012). postulated that A special feature of Lattice designs is that the number of treatments "v" is related to the block size "k" in the form $v = k^2$ or $v = k^3$ or $v = k(k+1)$ or $v = {k^2}/{2}$. Even though this limits the number of possible designs, Lattice Design represent an important class of designs. It can be broadly classified as square, circular, cubic and rectangular Lattice designs.

Srisuradetchai (2012) further maintained that A balanced square lattice design is similar to a balanced incomplete block design with k^2 treatments blocks with k runs per block and r = k + 1 replications. So, each replication has k blocks and contains every treatment. In this design, every pair of treatments occurs together once in the same incomplete block. This property holds for all plans having an odd number of treatments which is also a perfect square (e.g. 9, 25, 49, 81, 121, and 169 treatments). Let λ be an integer r number indicating how many times each treatment occurs together in same block, and the relationship among the number of treatments t, block size k, and number of replications r. Numerically, it is defined as λ = r (k − 1) = (t − 1). In balanced square lattice designs, r = k + 1 or $t = k^2$, implying λ = 1. Because the designs are balanced; all treatment differences have the same estimated variance or the same precision.

The number of replications required for balanced lattice becomes very large as the number of treatments increases. For this reason it is not usually practical to use balanced lattices for blocks with more than about seven units per block. In the interest of economy, then, the scientist is forced to accept a partially balanced design with fewer replications than would be required for full balance (Gaenamoe*t al. 2011)*.

The numbers of partially balanced square lattices are similar to balanced square lattices, but only some replications are selected (Srisurudetchai*et al.,* 2017).

Kling (2021) noted that

i. Simple Lattices: use first two replication from basic plan, 3x3 and 4x4 are no more precise than RCBD because error degree of freedom is too small
ii. Triple Lattices: use first three replications from basic plan. Possible for all squares from 3x3 to 13x13.
iii. Quadruple Lattices: use first four replications from basic plan do not exist for 6x6 and 10x10.

For example, for a4×4 partial lattices, we may have first, first two, first three or first four replications. With two replications, the partially Balanced Lattice Design is referred to as a simple lattice; with three replications, a triple lattice; with four replications, a quadruple lattice; and so on. In general, if the number of replication is m, it is called an m-ple Lattice.

2.0 Statement of Problem

In experimental design, one of the most important properties among others is the balance of the design. This property ensures that every treatment mean is estimated equally. In some designs, this property may not hold due to shortage of experimental materials. For example, if a Lattice design is to be balanced, then the number of replications should be $k+1$ where k the size of a block. As the number of the block increases, the number of replications also increases in order to achieve this balance, thereby stretching the number of treatment combinations; Partially Balanced Lattice Designs has numerous advantages including greater flexibility, efficient use of resources (evaluation of large number of treatments with smaller number of experimental units), time saving and effectiveness. It therefore becomes necessary to compare the Optimality Criteria of a Partially Balanced Lattice Designs with three associate classes among two, three and four replications using Design Based Optimality and hence promote its usage especially for experimentation. However, these challenges among other things motivated us to carry out a research on a Partially Balanced Lattice Designs with three associate classes and compare analysis of incidence matrixes of two, three and four replicates of the information matrix A-, D- and G-Optimality based Criteria to see if the number of replications determines the Optimality and Efficiency of the Designs.

The aim of this research is to compare Optimality Criteria for Partially Balanced Lattice Designs. The objectives of working with Lattice design can vary depending on the specific context but for this paper we want to investigate incidence matrix on A-, D- and G- Optimality Criteria in order to find out the criteria that is best and also to determine and compare the efficiency of the designs of incidence matrix of two, three and four replicates.

3.0Related Works

Many researchers have worked on a Partially Balanced Lattice Designs. These include Jyoti*et al.* (2016). On construction of Partially Balanced Incomplete Block Designs with two associate classes and the research revealed that D- criteria is more optimal and more efficient than A- D- and G- optimality criteria. Nenlat*et al.* (2017). On the study of Investigating the Optimality Criteria for a Partially Balanced Lattice Designs with two associate classes where the design matrices were both maximized and minimized with respect to information matrix and dispersion matrix in one, two and three replicates respectively. Based on their analysis, D- criteria show better results in all the replicates and the most recent research carried out by Salihu*et al.* (2021). On the study of Investigating the Optimality Criteria for Partially Balanced Lattice Designs with three Associate Classes, their analysis was based on incidence matrix of four replicates and it revealed that D- Criteria is more optimal and more efficient than A- and G- Criteria but the research failed to analyze the two and three replicates, it only duel on the four replicates let alone compare the optimality and efficiency of the designs to see if there is any different exists, the purpose is to compare the Optimality Criteria for Partially Balanced Lattice Designs with three Associate Classes of two, three and four replicates using A-, D- and G- Optimality Criteria.

4. Methodology

Design with two replications

$$X = \begin{pmatrix}
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0\\
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0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&1&0\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&1
\end{pmatrix}$$

Figure 1: incidence matrix of two replications

$$X'X = \begin{pmatrix}
4&0&0&0&0&0&0&0\\
0&4&0&0&0&0&0&0\\
0&0&4&0&0&0&0&0\\
0&0&0&4&0&0&0&0\\
0&0&0&0&4&0&0&0\\
0&0&0&0&0&4&0&0\\
0&0&0&0&0&0&4&0\\
0&0&0&0&0&0&0&4
\end{pmatrix}$$

Figure 2: Information matrix of two replications

$$(x'x)^{-1} = \begin{pmatrix}
0.25&0&0&0&0&0&0&0\\
0&0.25&0&0&0&0&0&0\\
0&0&0.25&0&0&0&0&0\\
0&0&0&0.25&0&0&0&0\\
0&0&0&0&0.25&0&0&0\\
0&0&0&0&0&0.25&0&0\\
0&0&0&0&0&0&0.25&0\\
0&0&0&0&0&0&0&0.25
\end{pmatrix}$$

Figure 3: inverse of Information matrix of two replications

$$A - Optimality = min_{xi=1,\dots,n} trace = sum(diag(x'x)^{-1}) = 2$$

$$D - Optimality = max_{xi,i=1,\dots,n}\, min_{x\in X}|x'x| = 65536$$

$$G - Optimality = min_{xi,i=1,\dots,n}\, max_{x\in X}\, x\,(x'x)^{-1}x' = 8$$

Designs with three replications

$$X = \begin{pmatrix}
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
1&0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0&0\\
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0&0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0&0\\
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0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix}$$

Figure 4: incidence matrix of three replications

$$X'X = \begin{pmatrix} 4&0&0&0&0&0&0&0&0&0&0&0\\ 0&4&0&0&0&0&0&0&0&0&0&0\\ 0&0&4&0&0&0&0&0&0&0&0&0\\ 0&0&0&4&0&0&0&0&0&0&0&0\\ 0&0&0&0&4&0&0&0&0&0&0&0\\ 0&0&0&0&0&4&0&0&0&0&0&0\\ 0&0&0&0&0&0&4&0&0&0&0&0\\ 0&0&0&0&0&0&0&4&0&0&0&0\\ 0&0&0&0&0&0&0&0&4&0&0&0\\ 0&0&0&0&0&0&0&0&0&4&0&0\\ 0&0&0&0&0&0&0&0&0&0&4&0\\ 0&0&0&0&0&0&0&0&0&0&0&4 \end{pmatrix}$$

Figure 5: Information matrix of three replications

$$(x'x)^{-1} = \begin{pmatrix} 0.25&0&0&0&0&0&0&0&0&0&0&0\\ 0&0.25&0&0&0&0&0&0&0&0&0&0\\ 0&0&0.25&0&0&0&0&0&0&0&0&0\\ 0&0&0&0.25&0&0&0&0&0&0&0&0\\ 0&0&0&0&0.25&0&0&0&0&0&0&0\\ 0&0&0&0&0&0.25&0&0&0&0&0&0\\ 0&0&0&0&0&0&0.25&0&0&0&0&0\\ 0&0&0&0&0&0&0&0.25&0&0&0&0\\ 0&0&0&0&0&0&0&0&0.25&0&0&0\\ 0&0&0&0&0&0&0&0&0&0.25&0&0\\ 0&0&0&0&0&0&0&0&0&0&0.25&0\\ 0&0&0&0&0&0&0&0&0&0&0&0.25 \end{pmatrix}$$

Figure 6: inverse of Information matrix of three replications

$$A - Optimality = min_{xi=1,\dots,n} trace = sum(diag(x'x)^{-1}) = 3$$

$$D - Optimality = max_{xi,i=1,\dots,n} min_{x\in X} |x'x| = 16777216$$

$$G - Optimality = min_{xi,i=1,\dots,n} max_{x\in X} x\ (x'x)^{-1} x' = 12$$

Design with four replications

$$X = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
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0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

Figure 7: incidence matrix of four replications

$$X'X = \begin{pmatrix}
4&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&4&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&4&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&4&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&4&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&4&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&4&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&4&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&4&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&4&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&4&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&4&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&4&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4
\end{pmatrix}$$

Figure 8: Information matrix of four replications

$$(x'x)^{-1} = \begin{pmatrix}
0.25&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0.25&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0.25&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0.25&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0.25&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0.25&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0.25&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0.25&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0.25&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0.25&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0.25&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0.25&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0.25&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0.25&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.25&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0.25
\end{pmatrix}$$

Figure 9: inverse of Information matrix of four replications

$$A - Optimality = min_{xi=1,\dots,n} trace = sum(diag(x'x)^{-1}) = 4$$

$$D - Optimality = max_{xi,i=1,\dots,n}\, min_{x\in X} |x'x| = 4.2950e+009$$

$$G - Optimality = min_{xi,i=1,\dots,n}\, max_{x\in X}\, x\,(x'x)^{-1}x' = 16$$

Note: P is the number of parameter and N is the number of treatments in the models.

4.2.2 Efficiency criteria

Equivalently, to obtain A-, D- and G- Efficiency criteria incidence matrices, it transposes and inverse of replication one, two, three and four above will be considered.

Efficiency with one replication

$$A - Efficiency = 100\left(\frac{k}{N * AOptimality}\right) = 100\left(\frac{4}{16 * 1}\right) = 25\%$$

$$D - Efficiency = 100\frac{(DOPtimality)^{1/k}}{N} = 100\frac{(256)^{1/4}}{16} = 25\%$$

$$\text{G} - \text{Efficiency} = 100\sqrt{\frac{P/N_D}{\delta^2{}_M}} = 100\sqrt{\frac{4/16}{1^2}} = 50\%$$

Where $k = block\ size = 4; N = number\ of\ treatments = 16$

Efficiency with two replications

$$A - Efficiency = 100\left(\frac{k}{N * AOptimality}\right) = 100\left(\frac{8}{32 * 2}\right) = 12.5\%$$

$$D - Efficiency = 100\frac{(D - Optimality)^{1/k}}{N} = 100\frac{(65536)^{1/8}}{32}$$

$$= 12.5\% \qquad G - \text{Efficiency} = 100\sqrt{\frac{P/N_D}{\delta^2{}_M}} = 100\sqrt{\frac{8/32}{2}}$$

$$= 35.36\% \qquad (4.28)$$

Where $k = block\ size = 8; N = number\ of\ treatments = 32$

Efficiency with three replications

$$A - Efficiency = 100\left(\frac{k}{N * AOptimality}\right) = 100\left(\frac{12}{48 * 3}\right) = 8.3333\%$$

$$D - Efficiency = 100\frac{(DOtimality)^{1/k}}{N} = 100\frac{(16777216)^{1/12}}{48} = 8.3333\%$$

$$G - \text{Efficiency} = 100\sqrt{\frac{P/N_D}{\delta^2{}_M}} = 100\sqrt{\frac{12/48}{3}} = 28.87\%$$

Where $k = block\ size = 12; N = number\ of\ treatments = 48$

Efficiency with four replications

$$A - Efficiency = 100\left(\frac{k}{N * AOptimality}\right) = 100\left(\frac{16}{64 * 4}\right) = 6.25\%$$

$$D - Efficiency = 100\frac{(D - Optimality)^{1/k}}{N} = 100\frac{(4.2950e + 009)^{1/16}}{64} = 6.25\%$$

$$G - \text{Efficiency} = 100\sqrt{\frac{P/N_D}{\delta^2{}_M}} = 100\sqrt{\frac{16/64}{4}} \ 25\%$$

Where $k = block\ size = 16; N = number\ of\ treatments = 64$

Table 4.3 Comparison of Optimality Criteria			
	Optimality Criteria		
No. of replication	A – optimality	D – optimality	G – optimality
Onereplication	1	256	4
tworeplications	2	65536	8
Threereplications	3	16777216	12
fourreplications	4	4.2950e+009	16

Table 4.3 Comparison of Efficiency Criteria			
	Efficiency Criteria		
No. of replication	**A –Efficiency**	**D –Efficiency**	**G – Efficiency**
Onereplication	0.2500	0.2500	0.5000
tworeplications	0.1250	0.1250	0.3535

Threereplications	0.0833	0.0833	0.2887
fourreplications	0.0625	0.0625	0.2500

5.0 Discussion of Results

Based on the analysis above D- optimality criteria produce better result follow by G- and A- criteria respectively, it translate that D-criteria is more optimal than other designs in all replications and it also reveal that the optimality of the design is dependent on the number of replications. In the other hand, A- and D- efficiency criteria have the same efficiency and more efficient than G- optimality criteria, it also show that the more, the replications the more the efficient of the designs. Despite the fact that A- and D- criteria have the same efficiency in all the replication under study but D- criteria turn to be more optimal than A- and G-criteria in all the replications and we conclude that that D-criteria is more optimal and more efficient than all other criteria. The result indicates that a D-optimality Criteria is the best suited for both Partially and Balanced Lattice design. This will be of interest to Agricultural researchers, engineers, pharmaceutical companies, environmentalist, computer analyst, social scientist and other researchers that may use lattice design in their experiment.

Recommendations

In the light of the above, it recommends here that:

iFor studies in lattice designs D- Optimality is better.

ii A standard based criterion was used for this research. However, modified and compound optimal designs are recommended for further research.

iiiExtension to four associate class designs recommended.

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