# NUMERICALLY IMPROVING THE RESULTS OF THE DOUBLE INTEGRALS WITH CONTINUOUS INTEGRATIONS USING AITKEN'S ACCELERATION WITH TRAPEZOIDAL RULE ON BOTH DIMENSIONS

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**ABSTRACT**

This research aims to improve the results of double integrals with continuous integrands numerically. The trapezoidal rule, which is one of Newton–Cotes formulas, has been used for the internal x and external y dimensions in the one-dimensional integrals' method. Thence, the results have been improved on both dimensions by using Aitken's acceleration to get a compound base in calculating double integrals that we have called the ATAT, where T refers to Trapezoidal, and A to Aitken's accelerating. Results have been quite accurate by using relatively few partials on x and y during a short time.

***Keywords :****double integrals, Aitken acceleration, the trapezoidal rule*

## INTRODUCTION

Calculating results of double integrals is not easy since the Integrand is based on two variables and since we are dealing with integration regions or surfaces and not with integration partials as it is the case with one-dimensional integrals. Therefore, it has been found necessary to find approximate values for these integrals. The importance of double integrals lies in calculating the area of surfaces, medium centers, moment of inertia for planar surfaces, and calculating the area beneath double integral surface.

This motivates many researchers to work on in double integrals' field. In 1973, Hans Schjar and Jacobsen [2] are two of the researchers who focused on calculating integrals with continuous integrates in the form: $f(x,y)=f_1(x)f_2(y)$, while some studied integrals with improper integrates but neglected the improperness, as Davis and Rabinowitz [3] did in 1975.

In 1984, Mohammed [7] treated integrals with continuous integrands by using a combined method of Equal-area rule and Romberg acceleration on the external dimension, and using Chaos rule on the internal dimension, which he then called the Romberg-Chaos method proven to be successful on many integrals with continuous integrands.

However, Al-Taee [4] in 2005 has used Equal-area rule and Romberg acceleration on the external dimension and the Simpson rule on the internal dimension. This experiment had had great results regarding accuracy, the number of partials used, and the consumed time to calculate those integrals.

As for researcher Dhia' [5] in 2009, the four methods of Romberg acceleration with the Simpson rule, and Romberg acceleration with the Equal-area rule, were used to give four ways to calculate values of double integrals with continuous integrands and those four ways are: $RM(RS)$, $RM(RM)$, $RS(RM)$ and $RS(RS)$. These four ways have proven to give great results regarding accuracy, number of partials used and consumed time; they have also proven to give the same results of equal levels of accuracy, number of partials used and consumed time as far as the integrals with continuous integrands in the integration area is concerned.

In 2011, Nasser [9] has added six ways to numerically calculate double integrals; such methods are combinations of the acceleration of both Romberg and Aitken and Equal-area and Simpson rules. Those ways are: $AMAM$, $AMRM$, $ASAS$, $ASRS$, $RSAS$ and $AMM$. When applied on double integrals with continuous integrates, these have shown great results regarding accuracy, number of partials used and the consumed time to achieve the result.

## II. EXPERIMENTAL

### STEP .1 :The ATAT Method

The ATAT method is a combined method of Aitken's acceleration with trapezoidal rule and on both dimensions (external **y** and internal **x**). Calculating the approximate value of double integrals in the combined method, is achieved as follows:

*If integration **I**is defined as:*

$$I = \int_c^d \int_a^b f(x, y)dx \quad dy \qquad (1)$$

*To find its approximate value, we put it as:*

$$I = \int_c^d F(y)dy \qquad (2)$$

*In which:*

$$F(y) = \int_a^b f(x, y)dx \qquad (3)$$

*The approximate integration on the external dimension **Y** for (2) using trapezoidal rule is:*

$$I = \frac{h}{2}\left[F(y_0) + 2\sum_{i=1}^{m-1} F(y_i) + F(y_m)\right] (4)$$

*Thence* $i = 1, 2, 3, ..., m, y_i = c + ih$ *and **m** is the number of partials dividing the period [c,d] and* $h = \frac{d-c}{m}$

*And to calculate the value of* $F(y_i)$ *approximately, values of* $y_i$ *are substituted in (3) to get:*

$$F(y_i) = \int_a^b f(x, y_i)dx \quad (5)$$

*And after applying the trapezoidal rule on the fifth equation in the internal dimension **x**, we get:*

$$F(y_i) = \frac{\bar{h}}{2}\left[f(x_0, y_i) + 2\sum_{j=1}^{n-1} f(x_j, y_i) + f(x_n, y_i)\right] (6)$$

*In which* $j = 1, 2, 3, ..., n, x_j = a + j\bar{h}$ *and **n** is the number of partials dividing the period [a,b] and* $\bar{h} = \frac{b-a}{n}$.

We will try to improve values of $F(y_i)$ by using Aitken's acceleration.

The resulting values of $F(y_i)$ are approximate values for integration (3) after applying Aitken's acceleration and replacing $F(y_i)$ for each $i = 1, 2, 3, ..., m$ in the fourth equation. We can apply Aitken's acceleration on these values in order to get integration values of (2) in the method of Aitken's acceleration with trapezoidal rule, which leads to finding out the value of integration (1) approximately.

As for dividing periods [a, b] and [c, d] into **m** and **n** of partials respectively, we use the following values when applying the ATAT method:

$m = 1, 2, 4, 8, \ldots$

$n = 1, 2, 4, 8, \ldots$

When we use $n = 1$ in trapezoidal rule, we calculate the value of integration (2) using Aitken's acceleration method on the values of trapezoidal rule. Then we use $n = 2$ and calculate the value of integration (2) in the same way until we achieve a value where the absolute error equals or is less than EPS1 on the external dimension **Y**.

Furthermore, in order to calculate the value of integration (2), we ought to find $F(y_1)$ by using Aitken's acceleration with the trapezoidal rule from integration (5). Substituting $m = 1$ in the fourth equation, $F(y_1)$ must be calculated by using the sixth equation when $n = 1$ when we use the trapezoidal rule on the internal integration. Thus, we then can substitute $n = 2$ and up until we reach a value where the absolute error is less than or equals a specific value called EPS (on the internal dimension).

Presumably, the needed value is found at $n = 8$. In this case, we set this approximate value of the integration in the table as a resulting value when $n = 8$ , $m = 1$.

However, if $m = 2$, we must calculate $F(y_1)$ and $F(y_2)$ for the fourth equation by applying the fifth equation when $n = 1$ then when $n = 2$ and so on.

Presumably, the error in $F(y_1)$ equals or is lesser than EPS when $n = 16$ and the absolute error in $F(y_2)$ equals or is lesser than $n = 64$. Then we set the approximate values of the first integration in the table, as resulted values, when $n = 64$ , $m = 2$. Particularly, setting the larger value for **n**, therefore **m>2**.

**STEP. 2; Aitken's Delta – Squared Process**

In 1926, Alexander Aitken (1895-1964)found a new method to accelerate the rate of Convergence of the calculated sequence to find out the output of a specific equation. To further clarify this method, we consider $\{x_n\}$ in which $\{x_n\} = \{x_1, x_2, \ldots x_k \ldots\}$ linearly converge into a final output $\beta$ .

*So*

$$\beta - x_{i+1} = C_i(\beta - x_i)$$

*In which*

$$|C_i| < 1$$

*And*

$$C_i \to C$$

*Note that* $C_i$ *is almost constant, so we may write:*

$$\beta - x_{i+1} \triangleq \bar{C}(\beta - x_i) \qquad (7)$$

*Whereas* $|\bar{C}| = C$ *and we note that:*

$$\frac{\beta - x_{i+2}}{\beta - x_{i+1}} \approx \frac{\beta - x_{i+1}}{\beta - x_i}$$

*Accordingly*

$$\beta \approx \frac{x_i x_{i+2} - x_{i+1}^2}{x_{i+2} - 2x_{i+1} + x_i} = x_{i+2} - \frac{(\Delta x_{i+1})^2}{\Delta^2 x_i} \qquad (8)$$

In which $\Delta x_i = (x_{i+1} - x_i)$ and $\Delta^2 x_i = x_{i+2} - 2x_{i+1} + x_i$

This is called Aitken's delta – Squared Process (Ralston [1])

When choosing $n$ from the sequence $\{x_n\}$ we can get $n-2$ from another sequence. In which $\{S\}$ converges to $\beta$ faster than $\{x_n\}$ as in:

$$s_{i+2} = x_{i+2} - \frac{(\Delta x_{i+1})^2}{\Delta^2 x_i} \qquad (9)$$

As $i = 1, 2, \ldots, n-2$

This process is accelerating the convergence of resulted values in anyway toward the final output $\beta$. For instance, if **n** was a value in trapezoidal rule, we can from there, get $n-2$ with Aitken's acceleration using equation (9), and then using Aitken's acceleration on the values $n-2$ to get $n-4$ values and we continue to get the needed accuracy (Mohammed [8] Nasser [9]).

Note: in the trapezoidal rule, we use $n = 1, 2, 4, 8, 16, \ldots\ldots$ (see Nasser [9])

**STEP .3 : Examples**

The integrand of integration $I = \int_1^2 \int_1^2 ln(x+y)dxdy$ is defined for each $(x, y) \in [1,2] \times [1,2]$ and its analytical valueis **1.0891386521** rounded to ten decimal numbers. And the form $F(y)$ for the above mentioned integration can be derived from form (3) by integrating it analytically for **x** to get:

$$F(y) = (2+y)ln(2+y) - (1+y)ln(1+y) - 1$$

Which is a continuous function in the period [1,2].

## III. RESULTS

Results derived from the ATAT method are listed in *table1*.

***Table .1. Calculating the Double Integral***

| *n* | *ATAT* | *m* |
|---|---|---|
| 32 | 1.7041118309 | 1 |
| 32 | 1.0866925506 | 2 |
| 32 | 1.0885202598 | 4 |
| 32 | 1.0891391322 | 8 |
| 32 | 1.0891386829 | 16 |
| 32 | 1.0891386520 | 32 |
| 32 | 1.0891386580 | 64 |
| 32 | 1.0891386518 | 128 |

*Note :Table1. Calculating the Double Integral* $I = \int_1^2 \int_1^2 ln(x+y)dxdy$ *by using the ATAT Method*

When $m = 128$ and $n = 32$, the value is correct for nine decimal numbers using $2^{12}$ partials with a difference of 3 absolute units in the tenth decimal.

To calculate the double integral $I = \int_1^2 \int_0^1 xe^{-(x+y)}dxdy$ numerically, the integrands are obviously defined for every $(x, y) \in [0,1] \times [1,2]$ and its analytical value is 0.0614477281 rounded to ten decimals. Then, the form $F(y)$ can be derived from form (3) when analytically integrated and resulted in: $F(y) = -2e^{-(y-1)} + e^{-y}$ which is a continuous function in the period [1, 2]. Results derived using the ATAT method are shown Table 2 below:

***Table .2 . Calculating the Double Integral***

| *n* | *ATAT* | *m* |
|---|---|---|
| 32 | 0.0843655839 | 1 |
| 64 | 0.0627225898 | 2 |
| 32 | 0.0617233396 | 4 |
| 32 | 0.0614473970 | 8 |
| 32 | 0.0614477079 | 16 |
| 32 | 0.0614477279 | 32 |
| 32 | 0.0614477281 | 64 |

*Note Table 2 Calculating the Double Integral* $I = \int_1^2 \int_0^1 xe^{-(x+y)}dxdy$ *using the ATAT method*

From Table 2, we conclude that when $m = 64$ and $n = 32$, the result of integration using the ATAT method is correct for at least ten decimals (similar to the analytical value) using $2^{11}$ partials. It is also noted in Table 2 that when $m = n = 32$ the value is correct for at least nine decimals with a difference of absolute two units in the tenth decimal.

Furthermore, the integrand of the double integral $I = \int_1^2\int_1^2 \ln(xy)dxdy$ defined for every $(x, y) \in [1,2]\times[1,2]$ and its analytical value, rounded to eleven decimals, is 0.77258872223. The form $F(y)$ then can be derived from analytically integrating form (3) to get: $F(y) = 2\ln 2y - \ln y - 1$ which is a continuous function in the period [1, 2]. Results derived by using the ATAT method are listed in Table 3.

***Table.3. calculating the double***

| *n* | *ATAT* | *m* |
|---|---|---|
| 64 | 1.27258872090 | 1 |
| 64 | 0.76231371079 | 2 |
| 64 | 0.76987999033 | 4 |
| 64 | 0.77259741554 | 8 |
| 64 | 0.77258932569 | 16 |
| 64 | 0.77258872081 | 32 |
| 64 | 0.77258872223 | 64 |

*Note Table3 calculating the double integral* $I = \int_1^2\int_1^2 \ln(xy)dxdy$ *using the ATAT method*

It is noted from table3 that when $m = 64$ and $n = 64$, the value of integration is correct for eleven decimals (similar to the analytical value) using $2^{12}$ partials.

Moreover, the double integral $I = \int_0^{\pi/2}\int_0^{\pi/2} \sin(x + y)dxdy$, is also continuous for every $(x, y) \in [0, \pi/2]\times[0, \pi/2]$ and its analytical value is 2, and the form $F(y)$ can be derived by analytically integrating form (3) to result: $F(y) = \cos y + \sin y$ which is a continuous function on the period $[0, \pi/2]$. Results derived using the ATAT method are listed in table4.

***Table. 4: Calculating the Double Integra***

| *n* | *ATAT* | *m* |
|---|---|---|
| 64 | 2.356194490 | 1 |
| 64 | 1.896118901 | 2 |
| 64 | 1.962894318 | 4 |
| 64 | 1.999933487 | 8 |
| 64 | 1.999995971 | 16 |
| 64 | 2.000000000 | 32 |

*Note Table 4 Calculating the Double Integra* $I = \int_0^{\pi/2}\int_0^{\pi/2} \sin(x + y)dxdy$ *using the ATAT method*

It is noticed IN Table 4 that when $m=32$ and $n=64$ the value of integration is correct when using the ATAT method is correct for at least nine decimals (similar to the analytical value) using $2^{11}$ partials.

The double integral's integrate $I=\int_0^{\ln 2}\int_0^{\ln 3} e^{(x+y)}dxdy$ is also continuous for every $(x,y)\in[0,\ln 3]\times[0,\ln 2]$ and its analytical value is 2. The form $F(y)$ can be derived from form (3) where $F(y)=2e^{y}$, which is a continuous function in the period; results of using the ATAT method are listed in Table 5.

***Table .5. Calculating the Double Integral***

| *n* | *ATAT* | *m* |
|---|---|---|
| 32 | 3.46573590594 | 1 |
| 32 | 2.01997891587 | 2 |
| 32 | 2.00484544814 | 4 |
| 32 | 1.99999749892 | 8 |
| 32 | 1.99999984257 | 16 |
| 32 | 2.00000000040 | 32 |
| 32 | 2.00000000573 | 64 |
| 32 | 2.00000000000 | 128 |

*Note Table 5 Calculating the Double Integral* $I=\int_0^{\ln 2}\int_0^{\ln 3} e^{(x+y)}dxdy$ *by using the ATAT method*

It is obvious in Table 5 that when $m=128$ and $n=32$, the numerical value of the integration using the ATAT method is correct at least for eleven decimals (similar to the analytical value) using $2^{12}$ partials.

## IV. DISCUSSION

Out of the results shown in the above tables and the chosen integrations, we conclude that when numerically calculating double integrals with continuous integrands by using the ATAT method on the two dimensions **x** and **y**, we get values that are identical to the analytical values and rounded to ten decimals or eleven in some continuous integrands and to nine decimals in other correct values of some continuous integrands. It is also noticed that the results rapidly approximate the real integration values by using relatively few partials.

## CONCLUSSIONS

the results have been improved on both dimensions by using Aitken's acceleration to get a compound base in calculating double integrals that we have called the ATAT, where T refers to Trapezoidal, and A to Aitken's accelerating. Results have been quite accurate by using relatively few partials on x and y during a short time.

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